2.1 Searching and algorithms

Algorithms

An ***algorithm*** is a sequence of steps for accomplishing a task. ***Linear search*** is a search algorithm that starts from the beginning of a list, and checks each element until the search key is found or the end of the list is reached.

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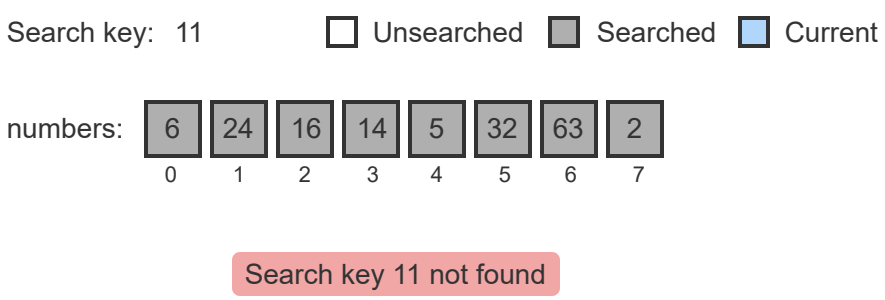
PARTICIPATION ACTIVITY 2.1.1: Linear search algorithm checks each element until key is found.



Animation content:

Static figure: An array named "numbers" is shown: [6, 24, 16, 14, 5, 32, 63, 2].

Step 1: Linear search starts at the first element and compares elements one-by-one.
The search key is set to 14. Numbers are highlighted as each is compared against the search key. At the end, searched elements 6, 24, and 16 are highlighted in gray. The current element 14 is highlighted in blue. Remaining unsearched elements 5, 32, 63, and 2 are highlighted in white.

Step 2: Linear search compares all elements if the search key is not present.
The search key is set to 11. Numbers are highlighted as each is compared against the search key. At the end, all elements have been searched, and so are highlighted in gray.

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Animation captions:

1. Linear search starts at the first element and compares elements one-by-one.
2. Linear search compares all elements if the search key is not present.

Figure 2.1.1: Linear search algorithm.

```
LinearSearch(numbers, numbersSize, key) {
   i = 0

   for (i = 0; i < numbersSize; ++i) {
      if (numbers[i] == key) {
         return i
      }
   }

   return -1 // not found
}

main() {
   numbers = {2, 4, 7, 10, 11, 32, 45, 87}
   NUMBERS_SIZE = 8
   i = 0
   key = 0
   keyIndex = 0

   print("NUMBERS: ")
   for (i = 0; i < NUMBERS_SIZE; ++i) {
      print(numbers[i] + " ")
   }
   printLine()

   print("Enter a value: ")
   key = getIntFromUser()

   keyIndex = LinearSearch(numbers, NUMBERS_SIZE, key)

   if (keyIndex == -1) {
      printLine(key + " was not found.")
   }
   else {
      printLine("Found " + key + " at index " + keyIndex + ".")
   }
}
```

```
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 10
Found 10 at index 3.
...
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 17
17 was not found.
```

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PARTICIPATION ACTIVITY 2.1.2: Linear search algorithm execution.

Given list: (20, 4, 114, 23, 34, 25, 45, 66, 77, 89, 11).

1) How many list elements will be compared to find 77 using linear search?

Check Show answer

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2) How many list elements will be checked to find the value 114 using linear search?

Check Show answer

3) How many list elements will be checked if the search key is not found using linear search?

Check Show answer

Algorithm runtime

An algorithm's ***runtime*** is the time the algorithm takes to execute. If each comparison takes 1 µs (1 microsecond), a linear search algorithm's runtime is up to 1 s to search a list with 1,000,000 elements, 10 s for 10,000,000 elements, and so on. Ex: Searching Amazon's online store, which has more than 200 million items, could require more than 3 minutes.

An algorithm typically uses a number of steps proportional to the size of the input. For a list with 32 elements, linear search requires at most 32 comparisons: 1 comparison if the search key is found at index 0, 2 if found at index 1, and so on, up to 32 comparisons if the search key is not found. For a list with N elements, linear search thus requires at most N comparisons. The algorithm is said to require "on the order" of N comparisons.

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PARTICIPATION ACTIVITY 2.1.3: Linear search runtime.

1) Given a list of 10,000 elements, and if each comparison takes 2 µs, what is the fastest possible runtime for linear search?

µs

Check Show answer

2) Given a list of 10,000 elements, and if each comparison takes 2 µs, what is the longest possible runtime for linear search?

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µs

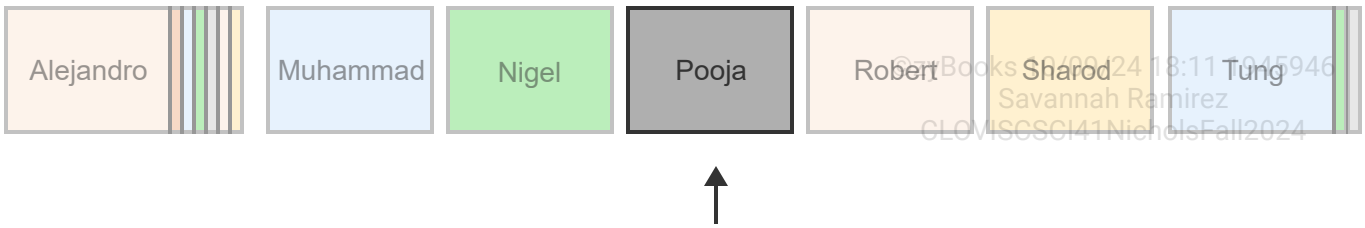
Check Show answer

2.2 Binary search

Linear search vs. binary search

Linear search may require searching all list elements, which can lead to long runtimes. For example, searching for a contact on a smartphone one-by-one from first to last can be time consuming. Because a contact list is sorted, a faster search, known as binary search, checks the middle contact first. If the desired contact comes alphabetically before the middle contact, binary search will then search the first half and otherwise the last half. Each step reduces the contacts that need to be searched by half.

PARTICIPATION ACTIVITY 2.2.1: Using binary search to search contacts on your phone.



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Animation content:

Static figure:
Contacts are shown: Alejandro, Muhammad, Nigel, Pooja, Robert, Sharod, and Tung. A list of hidden contacts are peeking out from behind Alejandro as well as Tung. An arrow, representing the contact checked during a search, is positioned under Pooja.

Step 1: Contacts are stored sorted by name. Searching for Pooja using a binary search starts by checking the middle contact.
The first, middle, and last contacts are revealed: Alejandro, Muhammad, and Nigel. An arrow is revealed under Muhammad, indicating that the middle contact is search first.

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Step 2: The middle contact is Muhammad. Pooja is alphabetically after Muhammad, so the binary search only searches the contacts after Muhammad. Only half the contacts now need to be searched.
The arrow moves from the middle contact, Muhammad, to the rest of the contacts starting with Nigel.

Step 3:Binary search continues by checking the middle element between Muhammad and the last contact.
Sharod is revealed as the middle contact between Nigel and the last contacts which is revealed to start with Tung.
The arrow moves to under Sharod.

Step 4: Pooja is before Sharod, so the search continues with only those contacts between Muhammad and Sharod, which is one fourth of the contacts.
The arrow moves backward to contacts between Nigel and Sharod.

Step 5: The middle element between Muhammad and Sharod is Pooja. Each step reduces the number of contacts to search by about half.
All contacts between Muhammad and Sharod are revealed. The contacts between Muhammad and Sharod are Nigel, Pooja, and Robert. Arrow moves to under Pooja.

Animation captions:

1. Contacts are stored sorted by name. Searching for Pooja using a binary search starts by checking the middle contact.
2. The middle contact is Muhammad. Pooja is alphabetically after Muhammad, so the binary search only searches the contacts after Muhammad. Only half the contacts now need to be searched.
3. Binary search continues by checking the middle element between Muhammad and the last contact.
4. Pooja is before Sharod, so the search continues with only those contacts between Muhammad and Sharod, which is one fourth of the contacts.

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5. The middle element between Muhammad and Sharod is Pooja. Each step reduces the number of contacts to search by about half.

PARTICIPATION ACTIVITY 2.2.2: Using binary search to search a contact list.

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A contact list is searched for Bob.
Assume the following contact list: (Amy, Bob, Chris, Holly, Ray, Sarah, Zoe)

1) What is the first contact searched?

Check Show answer

2) What is the second contact searched?

Check Show answer

Binary search algorithm

Binary search is a faster algorithm for searching a list if the list's elements are sorted and directly accessible (such as an array). Binary search first checks the middle element of the list. If the search key is found, the algorithm returns the matching location. If the search key is not found, the algorithm repeats the search on the remaining left sublist (if the search key was less than the middle element) or the remaining right sublist (if the search key was greater than the middle element).

PARTICIPATION ACTIVITY 2.2.3: Binary search efficiently searches sorted list by reducing the search space by half each iteration.

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Search key: 16 Unsearched Searched Current

numbers:	2	5	6	14	16	24	32	63
	0	1	2	3	4	5	6	7

16 == 16
Search key found at index 4

Animation content:

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Static figure:
A list named "numbers" contains 8 elements: [2, 5, 6, 14, 16, 24, 32, 63]. Index labels are shown under corresponding list elements starting from 0 to 7. A label "Search key: 16" is shown above the array.

Step 1: Elements with indices between low and high remain to be searched. The element at index 0, value 2, is labeled low, and the element at index 7, value 63, is labeled high.

Step 2: Search starts by checking the middle element.
Index named "mid" is computed as (high + low) // 2 = (7 + 0) // 2 = 3. The list element at index 3, 14, is compared with 16 and is not equal.

Step 3: If search key is greater than element, then only elements in right sublist need to be searched. 16 is greater than 14. The new low is calculated: low = mid + 1 = 4. low moves to the element at index 4, which is 16. high remains at index 7. mid at index 3 is grayed out to show it has already been searched. A new mid is calculated: mid = (high + low) // 2 = 5. mid moves to index 5, which has value 24.

Step 4: Each iteration reduces search space by half. Search continues until key found or search space is empty.
16 is less than 24, so high and mid get recalculated. high = mid - 1 = 4. mid = (4 + 4) / 2 = 4. high and mid move to index 4, and low is already at index 4. The list element at index 4 is 16, so the search key is found.

Animation captions:

1. Elements with indices between low and high remain to be searched.
2. Search starts by checking the middle element.
3. If search key is greater than element, then only elements in right sublist need to be searched.
4. Each iteration reduces search space by half. Search continues until key found or search space is empty.

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Figure 2.2.1: Binary search algorithm.

```
BinarySearch(numbers, numbersSize, key) {
   mid = 0
   low = 0
   high = numbersSize - 1

   while (high >= low) {
      mid = (high + low) / 2
      if (numbers[mid] < key) {
         low = mid + 1
      }
      else if (numbers[mid] > key) {
         high = mid - 1
      }
      else {
         return mid
      }
   }

   return -1 // not found
}

main() {
   numbers = { 2, 4, 7, 10, 11, 32, 45, 87 }
   NUMBERS_SIZE = 8
   i = 0
   key = 0
   keyIndex = 0

   print("NUMBERS: ")
   for (i = 0; i < NUMBERS_SIZE; ++i) {
      print(numbers[i] + " ")
   }
   printLine()

   print("Enter a value: ")
   key = getIntFromUser()

   keyIndex = BinarySearch(numbers, NUMBERS_SIZE, key)

   if (keyIndex == -1) {
      printLine(key + " was not found.")
   }
   else {
      printLine("Found " + key + " at index " + keyIndex + ".")
   }
}
```

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```
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 10
Found 10 at index 3.
...
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 17
17 was not found.
```

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PARTICIPATION ACTIVITY 2.2.4: Binary search algorithm execution.

Given list: (4, 11, 17, 18, 25, 45, 63, 77, 89, 114).

1) How many list elements will be checked to find the value 77 using binary search?

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Check Show answer

2) How many list elements will be checked to find the value 17 using binary search?

Check Show answer

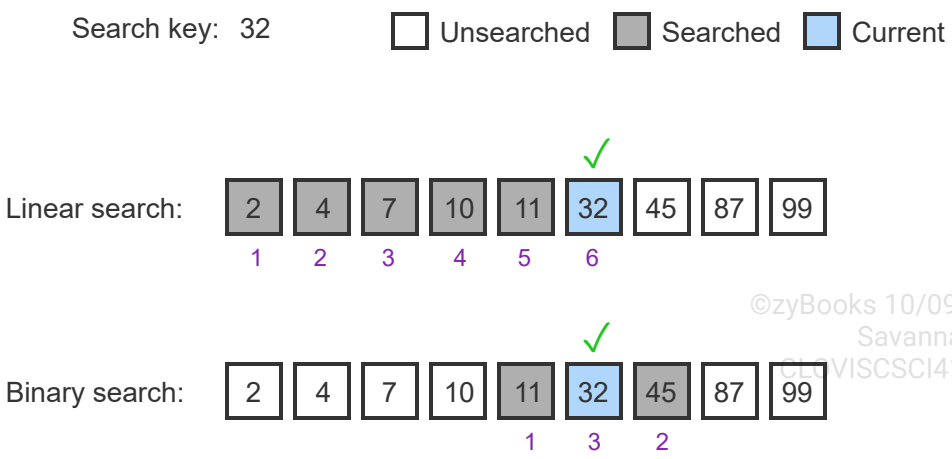
3) Given an array with 32 elements, how many list elements will be checked if the key is less than all elements in the list, using binary search?

Check Show answer

Binary search efficiency

Binary search is incredibly efficient in finding an element within a sorted array. During each iteration or step of the algorithm, binary search reduces the search space (i.e., the remaining elements to search within) by about half. The search terminates when the element is found or the search space is empty (element not found). For an N element array, the maximum number of steps required to reduce the search space to an empty subarray is $\lfloor log_2 N \rfloor + 1$. Ex: $\lfloor log_2 32 \rfloor + 1 = 6$.

PARTICIPATION ACTIVITY 2.2.5: Speed of linear search versus binary search.



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Animation content:

Static figure:
The label "Search key: 32" is shown.
An array labeled "Linear search" containing 9 elements it shown. The array elements are, 2, 4, 7, 10, 11, 32, 45, 87, 99 consecutively. Index labels are shown under the array elements from 2 to 32. The index labels are 1, 2, 3, 4, 5, 6 consecutively. Array elements 2, 4, 7, 10 and 11 have a gray background indicating that the elements have been searched. The array element 32 has a light blue background indicating that 32 is the current element being looked at. A green checkmark is shown above the element 32.
An array labeled "Binary search" containing 9 elements is shown. The array elements are, 2, 4, 7, 10, 11, 32, 45, 87, 99 consecutively. Array elements 11 and 45 have a gray background indicating that the elements have been searched. The array element 32 has a light blue background indicating that 32 is the current element being looked at. The index number 1 is shown under the element 11. The index number 3 is shown under the element 32. The index number 3 is shown under the element 45. A green checkmark is shown above the element 32.

Step 1: Binary search begins with the middle element. Each subsequent search reduces the search space by about half. Binary search finds a match after 3 comparisons.
Same array is shown for linear and binary search: [2, 4, 7, 10, 11, 32, 45, 87, 99]. First three comparisons are highlighted: 2, 4, and 7 for linear search, 11, 45, and 32 for binary search.

Step 2: Using linear search, a match is found after 6 comparisons. The binary search is more efficient, requiring only 3 comparisons.
Additional elements are highlighted for linear search: 10, 11, 32.

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Animation captions:

1. Binary search begins with the middle element. Each subsequent search reduces the search space by about half. Binary search finds a match after 3 comparisons.

2. Using linear search, a match is found after 6 comparisons. The binary search is more efficient, requiring only 3 comparisons.

If each comparison takes 1 µs (1 microsecond), a binary search algorithm's runtime is at most 20 µs to search a list with 1,000,000 elements, 21 µs to search 2,000,000 elements, 22 µs to search 4,000,000 elements, and so on. Ex: Searching Amazon's online store, which has more than 200 million items, requires less than 28 µs; up to 7,000,000 times faster than linear search.

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PARTICIPATION ACTIVITY 2.2.6: Linear and binary search efficiency.

1) Suppose a list of 1024 elements is searched with linear search. How many distinct list elements are compared against a search key that is less than all elements in the list?

elements

Check Show answer

2) Suppose a sorted list of 1024 elements is searched with binary search. How many distinct list elements are compared against a search key that is less than all elements in the list?

elements

Check Show answer

CHALLENGE ACTIVITY 2.2.1: Binary search.

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569966.3891892.qx3zqy7

Start

A foods list is searched for Cream using binary search.
Foods list: (Bread, Butter, Cheese, Chocolate, Coffee, Cream, Milk, Oatmeal, Rice, Tea)

What is the first food searched?

Pick

What is the second food searched?

Pick

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1 2 3 4 5

Check Next

2.3 Constant time operations

Constant time operations

In practice, designing an efficient algorithm aims to lower the amount of time that an algorithm runs. However, a single algorithm can always execute more quickly on a faster processor. Therefore, the theoretical analysis of an algorithm describes runtime in terms of number of constant time operations, not nanoseconds. A ***constant time operation*** is an operation that, for a given processor, always operates in the same amount of time, regardless of input values.

PARTICIPATION ACTIVITY 2.3.1: Constant time vs. non-constant time operations.

Constant time operations

```
x = 10      // assignment
y = 20      // assignment
a = 1000    // assignment
b = 2000    // assignment

z = x * y   // multiplication

c = a * b   // multiplication
```

Each multiplication completes in the same amount of time

Non-constant time operations

```
for (i = 0; i < x; i++) {
   sum += y
}
```

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```
// assume str1 is a string
// assume str2 is a string
str3 = concat(str1, str2)
```

Animation content:

Static figure: A code block labeled Constant time operations and two code blocks labeled Non-constant time operations are displayed.
Constant time operations:
Begin pseudocode:

```
x = 10      // assignment
y = 20      // assignment
a = 1000    // assignment
b = 2000    // assignment

z = x * y   // multiplication

c = a * b   // multiplication
```

End pseudocode.

First non-constant time operations code block:
Begin pseudocode:

```
for (i = 0; i < x; i++) {
   sum += y
}
```

End pseudocode.

Second non-constant time operations code block:
Begin pseudocode:

```
// assume str1 is a string
// assume str2 is a string
str3 = concat(str1, str2)
```

End pseudocode.

Step 1: Statements x = 10, y = 20, a = 1000, and b = 2000 assign values to fixed-size integer variables. Each assignment is a constant time operation.
Step 2: A CPU multiplies values 10 and 20 at the same speed as 1000 and 2000. Multiplication of fixed-size integers is a constant time operation. The lines of code, z = x * y, c = a * b, are highlighted and the text, each multiplication completes in the same amount of time, is displayed.
Step 3: A loop that iterates x times, adding y to a sum each iteration, will take longer if x is larger. The loop is not constant time.
Step 4: String concatenation is another common operation that is not constant time, because more characters must be copied for larger strings.

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Animation captions:

1. Statements x = 10, y = 20, a = 1000, and b = 2000 assign values to fixed-size integer variables. Each assignment is a constant time operation.
2. A CPU multiplies values 10 and 20 at the same speed as 1000 and 2000. Multiplication of fixed-size integers is a constant time operation.
3. A loop that iterates x times, adding y to a sum each iteration, will take longer if x is larger. The loop is not constant time.
4. String concatenation is another common operation that is not constant time, because more characters must be copied for larger strings.

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PARTICIPATION ACTIVITY 2.3.2: Constant time operations.

1) The statement below that assigns x with y is a constant time operation.

```
y = 10
x = y
```

- ○ True
- ○ False

2) A loop is never a constant time operation.

- ○ True
- ○ False

3) The 3 constant time operations in the code below can collectively be considered 1 constant time operation.

```
x = 26.5
y = 15.5
z = x + y
```

- ○ True
- ○ False

Identifying constant time operations

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The programming language being used, as well as the hardware running the code, both affect what is and what is not a constant time operation. Ex: Most modern processors perform arithmetic operations on integers and floating point values at a fixed rate that is unaffected by operand values. Part of the reason for this is that the floating point and integer values have a fixed size. The table below summarizes operations that are generally considered constant time operations.

Table 2.3.1: Common constant time operations.

Operation	Example

Addition, subtraction, multiplication, and division of fixed size integer or floating point values.	`w = 10.4`
	`x = 3.4`
	`y = 2.0`
	`z = (w - x) / y`

Assignment of a reference, pointer, or other fixed size data value.	`x = 1000`
	`y = x`
	`a = true`
	`b = a`

Comparison of two fixed size data values.	`a = 100`
	`b = 200`
	`if (b > a) {`
	`  ...`
	`}`

Read or write an array element at a particular index.	`x = arr[index]` `arr[index + 1] = x + 1`

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PARTICIPATION ACTIVITY 2.3.3: Identifying constant time operations.

1) In the code below, suppose str1 is a pointer or reference to a string. The code only executes in constant time if the assignment copies the pointer/reference, and not all the characters in the string.

```
str2 = str1
```

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- ○ True
- ○ False

2) Certain hardware may execute division more slowly than multiplication, but both may still be constant time operations.

- True
- False

3) The hardware running the code is the only thing that affects what is and what is not a constant time operation.

- True
- False

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2.4 Growth of functions and complexity

Upper and lower bounds

An algorithm with runtime complexity T(N) has a lower bound and an upper bound.

- ***Lower bound***: A function f(N) that is ≤ the best case T(N), for all values of N ≥ 1.
- ***Upper bound***: A function f(N) that is ≥ the worst case T(N), for all values of N ≥ 1.

Given a function T(N), an infinite number of lower bounds and upper bounds exist. Ex: If an algorithm's best case runtime is T(N) = 5N + 4, then subtracting any nonnegative integer yields a lower bound: 5N + 3, 5N + 2, and so on. So two additional criteria are commonly used to choose a preferred upper or lower bound. The preferred bound:

1. is a single-term polynomial and
2. bounds T(N) as tightly as possible.

Upper and lower bounds in the context of runtime complexity

This section presents upper and lower bounds specifically in the context of algorithm complexity analysis. The constraint N ≥ 1 is included because of the assumption that every algorithm presented in this book operates on a dataset with at least 1 item.

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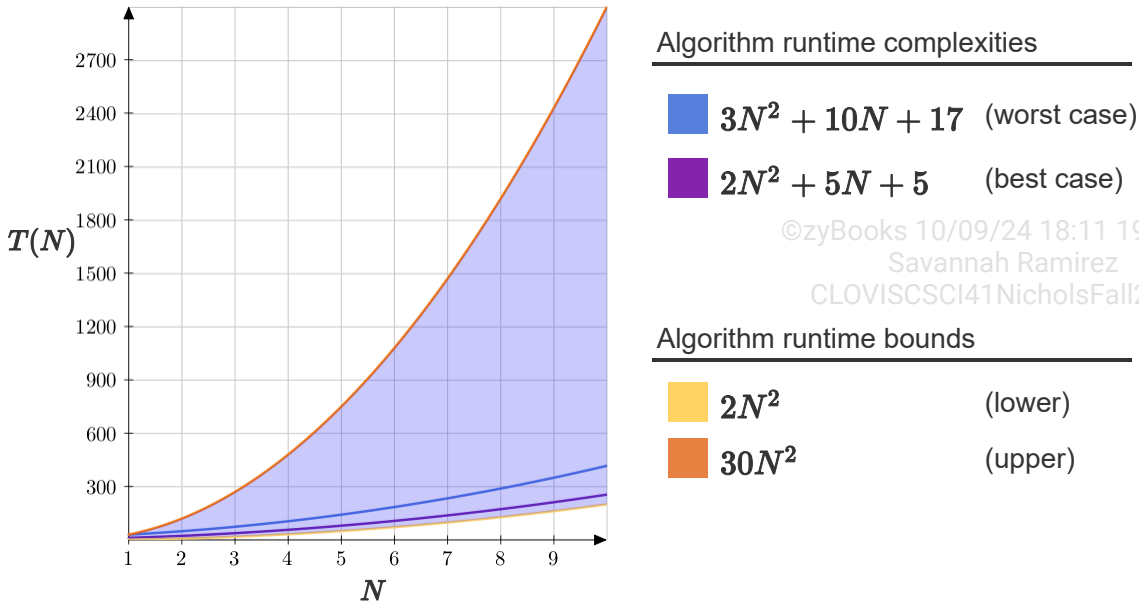
PARTICIPATION ACTIVITY 2.4.1: Upper and lower bounds with N_0 = 1.



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Animation content:

Static figure:
A graph is shown with N as the independent x-axis and T(N) as the dependent y-axis. Window is from 1 to 9 on the x-axis and from 0 to 3000 on the y-axis. 4 lines are shown on the graph representing the functions T(N) = 3N^2 + 10N + 17 (worst case), T(N) = 2N^2 + 5N + 5 (best case), 2N^2 (lower), and 30 N^2 (upper). The first two formulas are shown under the title "Algorithm runtime complexities". The second two formulas are shown under the title "Algorithm runtime bounds".

Step 1: A graph is shown with N as the independent x-axis and T(N) as the dependent y-axis. Window is from 1 to 9 on the x-axis and from 0 to 3000 on the y-axis. Functions T(N) = 3N^2 + 10N + 17 (worst case) and T(N) = 2N^2 + 5N + 5 (best case) are graphed and the formulas are revealed under the title "Algorithm runtime complexities".

Step 2: The label "Three-term polynomial" appears beneath the best case T(N). Three arrows appear to point out the polynomial's three terms: 2N^2, 5N, and 5.

Step 3: The lower bound, 2N^2, is graphed. The lower bound is below both T(N)s for all N values in the window.

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Step 4: Text appears: "Upper bound: CN^2 ≥ 3N^2 + 10N + 17 for all N ≥ 1".

Step 5: Text appears, describing that if N = 1 then 3N^2 = 3 and 3N^2 + 10N + 17 = 30. Since 3 is not ≥ 30, 3N^2 does not work as an upper bound.

Animation captions:

1. An algorithm's worst and best case runtimes are represented by the blue and purple curves, respectively.
2. The best case expression itself is a lower bound, but is a polynomial with three terms: $2N^2$, $5N$, and 5. A single-term polynomial would provide a simpler picture.
3. $2N^2$, shown in yellow, is a lower bound. The lower bound is less than or equal to the best case T(N) for all N ≥ 1.
4. The worst case's highest power of N is N^2. So the upper bound must be some constant C times N^2 such that $CN^2 \geq 3N^2 + 10N + 17$ for all N ≥ 1.
5. $3N^2$ does not work. Ex: When $N = 1$, $3N^2 = 3$ and $3N^2 + 10N + 17 = 30$.
6. The lowest C that satisfies requirements is 30. $30N^2$ is greater than or equal to the worst case T(N) for all N ≥ 1. So $30N^2$, shown in orange, is an upper bound.
7. Together, the upper and lower bounds enclose all possible runtimes for this algorithm.

PARTICIPATION ACTIVITY 2.4.2: Upper and lower bounds with N_0 = 1.

Suppose an algorithm's best case runtime complexity is $T(N) = 3N + 6$, and the algorithm's worst case runtime is $T(N) = 5N^2 + 7N$.

1) The algorithm has ____.

- only one possible lower bound
- multiple, but finite, lower bounds
- an infinite number of lower bounds

2) Which is the preferred lower bound?

- $3N + 6$
- $3N$
- N

3) $7N^2$ is ____ for the algorithm.

- an upper bound
- a lower bound
- neither a lower bound nor an upper bound

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4) Which function is an upper bound for the algorithm?

- $12N^2$
- $5N^2$
- $7N$

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Growth rates and asymptotic notations

An additional simplification can factor out the constant from a bounding function, leaving a function that categorizes the algorithm's growth rate. Ex: Instead of saying that an algorithm's runtime function has an upper bound of $30N^2$, the algorithm could be described as having a worst case growth rate of N^2. ***Asymptotic notation*** is the classification of runtime complexity that uses functions that indicate only the growth rate of a bounding function. Three asymptotic notations are commonly used in complexity analysis:

- ***O notation*** provides a growth rate for an algorithm's upper bound.
- ***Ω notation*** provides a growth rate for an algorithm's lower bound.
- ***Θ notation*** provides a growth rate that is both an upper and lower bound.

Table 2.4.1: Notations for algorithm complexity analysis.

Notation	General form	Meaning
O	$T(N) = O(f(N))$	A positive constant c exists such that, for all N ≥ 1, $T(N) \leq c * f(N)$.
Ω	$T(N) = \Omega(f(N))$	A positive constant c exists such that, for all N ≥ 1, $T(N) \geq c * f(N)$.
Θ	$T(N) = \Theta(f(N))$	$T(N) = O(f(N))$ and $T(N) = \Omega(f(N))$.

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PARTICIPATION ACTIVITY 2.4.3: Asymptotic notations.

Suppose $T(N) = 2N^2 + N + 9$.

1) $T(N) = O(N^2)$

- True
- False

2) $T(N) = \Omega(N^2)$

- True
- False

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3) $T(N) = \Theta(N^2)$

- True
- False

4) $T(N) = O(N^3)$

- True
- False

5) $T(N) = \Omega(N^3)$

- True
- False

2.5 O notation

Big O notation

Big O notation is a mathematical way of describing how a function (running time of an algorithm) generally behaves in relation to the input size. In Big O notation, all functions that have the same growth rate (as determined by the highest order term of the function) are characterized using the same Big O notation. In essence, all functions that have the same growth rate are considered equivalent in Big O notation.

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Given a function that describes the running time of an algorithm, the Big O notation for that function can be determined using the following rules:

1. If f(N) is a sum of several terms, the highest order term (the one with the fastest growth rate) is kept and others are discarded.

2. If f(N) has a term that is a product of several factors, all constants (those that are not in terms of N) are omitted.

PARTICIPATION ACTIVITY 2.5.1: Determining Big O notation of a function.

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Algorithm steps: $5 + 13 \cdot N + 7 \cdot N^2$

Big O notation: $O(5 + 13 \cdot N + 7 \cdot N^2) = O(7 \cdot N^2)$

$= O(N^2)$ Pronounced: "Oh N squared"

Rule 1: If f(N) is a sum of several terms, the highest order term is kept and others are discarded.

Rule 2: If f(N) has a term that is a product of several factors, all constants are omitted.

Animation content:

Static figure: The following text is displayed:
Algorithm steps: 5 + 13·N + 7·N sup 2
Big O notation: O(5 + 13·N + 7·N sup 2)
Step 1: Determine a function that describes the running time of the algorithm, and then compute the Big O notation of that function.
Step 2: Apply rules to obtain the Big O notation of the function. The text, Rule 1: If f(N) is a sum of several terms, the highest order term
is kept and others are discarded., is displayed. The text, 7·N sup 2, is highlighted and Big O notation: now shows O(5 + 13·N + 7·N sup 2) = O(7·N sup 2). The text, Rule 2: If f(N) has a term that is a product of several factors, all constants are omitted., is displayed and the equation, = O(N sup 2), appears under = O(7·N sup 2).
Step 3: All functions with the same growth rate are considered equivalent in Big O notation.

Animation captions:

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1. Determine a function that describes the running time of the algorithm, and then compute the Big O notation of that function.
2. Apply rules to obtain the Big O notation of the function.
3. All functions with the same growth rate are considered equivalent in Big O notation.

PARTICIPATION ACTIVITY 2.5.2: Big O notation.

1) Which of the following Big O notations is equivalent to O(N+9999)?

- O(1)
- O(N)
- O(9999)

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2) Which of the following Big O notations is equivalent to O(734·N)?

- O(N)
- O(734)
- O($734 \cdot N^2$)

3) Which of the following Big O notations is equivalent to O($12 \cdot N + 6 \cdot N^3 + 1000$)?

- O(1000)
- O(N)
- O(N^3)

Big O notation of composite functions

The following rules are used to determine the Big O notation of composite functions: c denotes a constant

Figure 2.5.1: Rules for determining Big O notation of composite functions.

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Composite function	Big O notation
c · O(f(N))	O(f(N))
c + O(f(N))	O(f(N))

g(N) · O(f(N))	O(g(N) · f(N))
g(N) + O(f(N))	O(g(N) + f(N))

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PARTICIPATION ACTIVITY 2.5.3: Big O notation for composite functions.

Determine the simplified Big O notation.

1) $10 \cdot O(N^2)$

- O(10)
- $O(N^2)$
- $O(10 \cdot N^2)$

2) $10 + O(N^2)$

- O(10)
- $O(N^2)$
- $O(10 + N^2)$

3) $3 \cdot N \cdot O(N^2)$

- $O(N^2)$
- $O(3 \cdot N^2)$
- $O(N^3)$

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4) $2 \cdot N^3 + O(N^2)$

- $O(N^2)$
- $O(N^3)$

- O($N^2 + N^3$)

5) $5 \cdot log_{10} N$

- O($5 \cdot logN$)
- O($log_{10}\ N$)
- O($log\ N$)

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Runtime growth rate

One consideration in evaluating algorithms is that the efficiency of the algorithm is most critical for large input sizes. Small inputs are likely to result in fast running times because N is small, so efficiency is less of a concern. The table below shows the runtime to perform f(N) instructions for different functions f and different values of N. For large N, the difference in computation time varies greatly with the rate of growth of the function f. The data assumes that a single instruction takes 1 µs to execute.

Table 2.5.1: Growth rates for different input sizes.

Function	N = 10	N = 50	N = 100	N = 1000	N = 10000	N = 100000
$log_2 N$	3.3 µs	5.65 µs	6.6 µs	9.9 µs	13.3 µs	16.6 µs
N	10 µs	50 µs	100 µs	1000 µs	10 ms	100 ms
$Nlog_2 N$	.03 ms	.28 ms	.66 ms	.0099 s	.132 s	1.66 s
N^2	.1 ms	2.5 ms	10 ms	1 s	100 s	2.7 hours
N^3	1 ms	.125 s	1 s	16.7 min	11.57 days	31.7 years
2^N	.001 s	35.7 years	> 1000 years			

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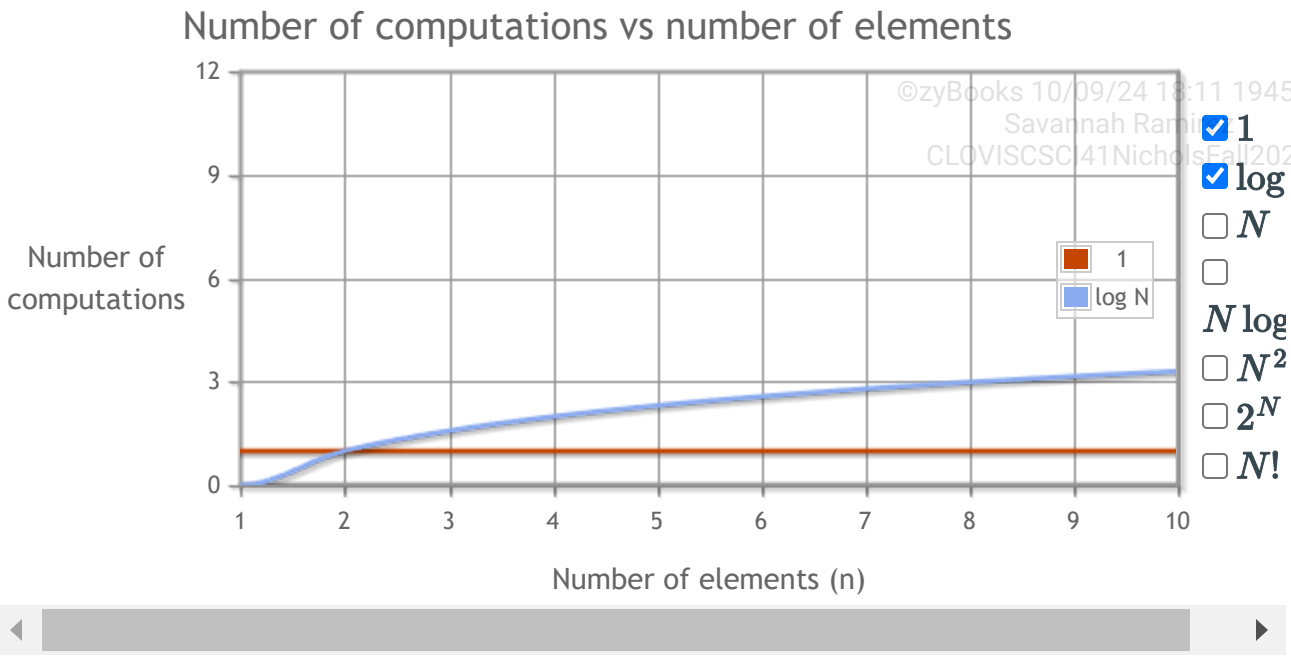
The interactive tool below illustrates graphically the growth rate of commonly encountered functions.

PARTICIPATION ACTIVITY 2.5.4: Computational complexity graphing tool.



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Common Big O complexities

Many commonly used algorithms have running time functions that belong to one of a handful of growth functions. These common Big O notations are summarized in the following table. The table shows the Big O notation, the common word used to describe algorithms that belong to that notation, and an example with source code. Clearly, the best algorithm is one that has constant time complexity. Unfortunately, not all problems can be solved using constant complexity algorithms. In fact, in many cases, computer scientists have proven that certain types of problems can only be solved using quadratic or exponential algorithms.

Figure 2.5.2: Runtime complexities for various code examples.

Notation	Name	Example pseudocode
O(1)	Constant	`FindMin(x, y) { if (x < y) { return x } else { return y } }`

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O(log N)	Logarithmic	`BinarySearch(numbers, N, key) {`
		`  mid = 0`
		`  low = 0`
		`  high = N - 1`
		
		`  while (high >= low) {`
		`  mid = (high + low) / 2`
		`  if (numbers[mid] < key) {`
		`  low = mid + 1`
		`  }`
		`  else if (numbers[mid] > key) {`
		`  high = mid - 1`
		`  }`
		`  else {`
		`  return mid`
		`  }`
		`  }`
		
		`  return -1  // not found`
		`}`

O(N)	Linear	`LinearSearch(numbers, numbersSize, key) {`
		`  for (i = 0; i < numbersSize; ++i) {`
		`  if (numbers[i] == key) {`
		`  return i`
		`  }`
		`  }`
		
		`  return -1 // not found`
		`}`

O(N log N)	Linearithmic	`MergeSort(numbers, i, k) {`
		`  j = 0`
		`  if (i < k) {`
		`  j = (i + k) / 2  // Find midpoint`
		
		`  MergeSort(numbers, i, j)  // Sort left part`
		`  MergeSort(numbers, j + 1, k) // Sort right part`
		`  Merge(numbers, i, j, k)  // Merge parts`
		`  }`
		`}`

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$O(N^2)$	Quadratic	`SelectionSort(numbers, numbersSize) {`
		`  for (i = 0; i < numbersSize; ++i) {`
		`  indexSmallest = i`
		`  for (j = i + 1; j < numbersSize; ++j) {`
		`  if (numbers[j] < numbers[indexSmallest])`
		`{`
		`  indexSmallest = j`
		`  }`
		`  }`
		`  temp = numbers[i]`
		`  numbers[i] = numbers[indexSmallest]`
		`  numbers[indexSmallest] = temp`
		`  }`
		`}`

$O(c^N)$	Exponential	`Fibonacci(N) {`
		`  if ((1 == N) || (2 == N)) {`
		`  return 1`
		`  }`
		`  return Fibonacci(N-1) + Fibonacci(N-2)`
		`}`

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PARTICIPATION ACTIVITY 2.5.5: Big O notation and growth rates.

1) O(5) has a _____ runtime complexity.
   - constant
   - linear
   - exponential

2) O(N log N) has a _____ runtime complexity.
   - constant
   - linearithmic
   - logarithmic

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3) $O(N + N^2)$ has a _____ runtime complexity.
   - linear-quadratic
   - exponential
   - quadratic

4) A linear search has a ____ runtime complexity.

- O(log N)
- O(N)
- O(N^2)

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5) A selection sort has a ____ runtime complexity.

- O(N)
- O(N log N)
- O(N^2)

2.6 Algorithm analysis

Worst-case algorithm analysis

To analyze how runtime of an algorithm scales as the input size increases, we first determine how many operations the algorithm executes for a specific input size, N. Then, the big-O notation for that function is determined. Algorithm runtime analysis often focuses on the worst-case runtime complexity. The ***worst-case runtime*** of an algorithm is the runtime complexity for an input that results in the longest execution. Other runtime analyses include best-case runtime and average-case runtime. Determining the average-case runtime requires knowledge of the statistical properties of the expected data inputs.

PARTICIPATION ACTIVITY 2.6.1: Runtime analysis: Finding the max value.

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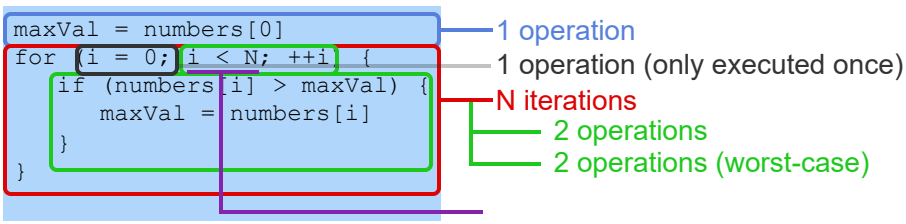

1 final comparison operation

$$f(N) = 1 + 1 + 1 + N(2 + 2)$$
$$= 3 + 4N$$
$$= O(N)$$

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Static figure:
Begin pseudo code:

```
maxVal = numbers[0]
for (i = 0; i < N; ++i) {
  if (numbers[i] > maxVal) {
    maxVal = numbers[i]
  }
}
```

End pseudo code.
The pseudo code block is shown with different parts highlighted by various boxes. The first line of code, "maxVal = numbers[0] " is highlighted in a blue box labeled "1 operation". Entire for loop, "for (i = 0; I < N; ++i) {
  if (numbers[i] > maxVal) {
    maxVal = numbers[i]
  }
}"
is highlighted in a red box labeled "N iterations". The For loop's "i = 0;" statement is highlighted in a black box labeled "1 operation (only executed once)". The For loop's other two statements, "i < N;" and "++i" are highlighted in a green box. The For loop's "i < N;" statement also has a purple underline with corresponding purple label "1 final comparison operation". The For loop's body, "if (numbers[i] > maxVal) {
    maxVal = numbers[i]
  }" is also highlighted in a green box, labeled "2 operations" for the first two statements and "2 operations (worst-case)" for the body's statements, respectively.

Text labels below code block shows the time complexity computation:
f(N) = 1 + 1 + 1 + N (2 + 2)
  = 3 + 4N
  = O(N)

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Step 1: Runtime analysis determines the total number of operations. Operations include assignment, addition, comparison, etc.
The pseudo code block appears. The first line of code, "maxVal = numbers[0] " is highlighted in a blue box labeled "1 operation".

Step 2: The for loop iterates N times, but the for loop's initial expression i = 0 is executed once.

Entire For loop, "for (i = 0; i < N; ++i) {
  if (numbers[i] > maxVal) {
    maxVal = numbers[i]
  }
}"
is highlighted in a red box labeled "N iterations". Next, the For loop's "i = 0;" statement is highlighted in a black box labeled "1 operation (only executed once)".

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Step 3: For each loop iteration, the increment and comparison expressions are each executed once. In the worst case, the if's expression is true, resulting in 2 operations.
The For loop's other two statements, "i < N;" and "++i" are highlighted in a green box labeled, "2 operations". Next, the For loop's body, "if (numbers[i] > maxVal) {
    maxVal = numbers[i]
  }" is highlighted in a green box, labeled "2 operations (worst-case)".

Step 4: One additional comparison is made before the loop ends.
A purple underline with corresponding purple label "1 final comparison operation" is shown under the For loop's "i < N;" statement.

Step 5: The function f(N) specifies the number of operations executed for input size N. The Big O notation for the function is the algorithm's worst-case runtime complexity.
Text labels are revealed below code block which shows the time complexity computation:
"f(N) = 1 + 1 + 1 + N (2 + 2)
  = 3 + 4N
  = O(N) "

Animation captions:

1. Runtime analysis determines the total number of operations. Operations include assignment, addition, comparison, etc.
2. The for loop iterates N times, but the for loop's initial expression i = 0 is executed once.
3. For each loop iteration, the increment and comparison expressions are each executed once. In the worst-case, the if's expression is true, resulting in 2 operations.
4. One additional comparison is made before the loop ends.
5. The function f(N) specifies the number of operations executed for input size N. The big-O notation for the function is the algorithm's worst-case runtime complexity.

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PARTICIPATION ACTIVITY 2.6.2: Worst-case runtime analysis.

1) Which function best represents the number of operations in the worst-case?

```
i = 0
sum = 0
while (i < N) {
   sum = sum + numbers[i]
   ++i
}
```

- f(N) = 3N + 2
- f(N) = 3N + 3
- f(N) = 2 + N (N + 1)

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2) What is the big-O notation for the worst-case runtime?

```
negCount = 0
for(i = 0; i < N; ++i) {
   if (numbers[i] < 0 ) {
      ++negCount
   }
}
```

- f(N) = 2 + 4N + 1
- O(4N + 3)
- O(N)

3) What is the big-O notation for the worst-case runtime?

```
for (i = 0; i < N; ++i) {
   if ((i % 2) == 0) {
      outVal[i] = inVals[i] * i
   }
}
```

- O(1)
- O($\frac{N}{2}$)
- O(N)

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4) Assuming nVal is an integer, what is the big-O notation for the worst-case runtime?

```
nVal = N
steps = 0
while (nVal > 0) {
   nVal = nVal / 2
   steps = steps + 1
}
```

- O(log N)
- O($\frac{N}{2}$)
- O(N)

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5) What is the big-O notation for the *best-case* runtime?

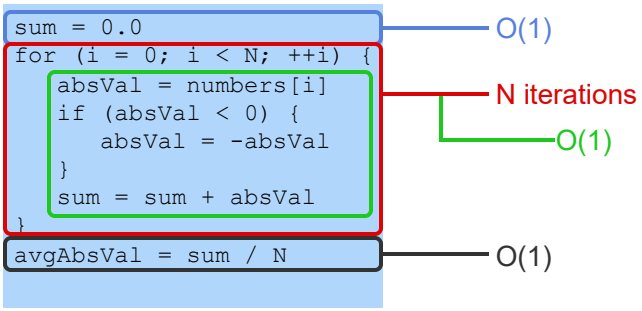
```
i = 0
belowThresholdSum = 0.0
belowThresholdCount = 0
while (i < N && numbers[i] <=
threshold) {
   belowThresholdCount += 1
   belowThresholdSum +=
numbers[i]
   i += 1
}
avgBelow = belowThresholdSum /
belowThresholdCount
```

- O(1)
- O(N)

Counting constant time operations

For algorithm analysis, the definition of a single operation does not need to be precise. An operation can be any statement (or constant number of statements) that has a constant runtime complexity, O(1). Since constants are omitted in big-O notation, any constant number of constant time operations is O(1). So, precisely counting the number of constant time operations in a finite sequence is not needed. Ex: An algorithm with a single loop that execute 5 operations before the loop, 3 operations each loop iteration, and 6 operations after the loop would have a runtime of f(N) = 5 + 3N + 6, which can be written as O(1) + O(N) + O(1) = O(N). If the number of operations before the loop was 100, the big-O notation for those operations is still O(1).

PARTICIPATION ACTIVITY 2.6.3: Simplified runtime analysis: A constant number of constant time operations is O(1).



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O(1) + NO(1) + O(1) = O(N)

Animation content:

Static figure: Begin code:
sum = 0.0
for (i = 0; i < N; ++i) {
  absVal = numbers[i]
  if (absVal < 0) {
    absVal = -absVal
  }
  sum = sum + absVal
}
double avgAbsVal = sum / N
End code.
The code block is shown with different parts highlighted by various boxes. The first line of code, "sum = 0.0", is highlighted in a blue box, labeled "O(1)". The For loop,
"for (i = 0; i < N; ++i) {
  absVal = numbers[i]
  if (absVal < 0) {
    absVal = -absVal
  }
  sum = sum + absVal;
}",
 is highlighted in a red box, labeled "N iterations". The For loop's body,
" absVal = numbers[i]
  if (absVal < 0) {
    absVal = -absVal
  }
  sum += absVal;",
 is encased in the red box and also highlighted in a green box, labeled "O(1)". The last line of code, "
avgAbsVal = sum / N", is highlighted in a black box, labeled "O(1)".
Text is shown below the code block indicating the time complexity calculation: O(1) + N * O(1) +

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O(1) = O(N).

Step 1: Constants are omitted in Big O notation, so any constant number of constant time operations is O(1). The blue box, green box, and black box are drawn.

Step 2: The for loop iterates N times. Big O complexity can be written as a composite function and simplified.
The red box containing the for loop is highlighted and is labeled "N iterations". Next, the time complexity calculation of the code is constructed from the labels from each highlighted area, "O(1) + N * O(1) + O(1) = O(N)".

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Animation captions:

1. Constants are omitted in big-O notation, so any constant number of constant time operations is O(1).
2. The for loop iterates N times. Big-O complexity can be written as a composite function and simplified.

PARTICIPATION ACTIVITY 2.6.4: Constant time operations.

1) A for loop of the form `for (i = 0; i < N; ++i) {}` that does not have nested loops or function calls, and does not modify i in the loop will always have a complexity of O(N).

   - True
   - False

2) The complexity of the algorithm below is O(1).

```
if (timeHour < 6) {
   tollAmount = 1.55
}
else if (timeHour < 10) {
   tollAmount = 4.65
}
else if (timeHour < 18) {
   tollAmount = 2.35
}
else {
   tollAmount = 1.55
}
```

   - True

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- False

3) The complexity of the algorithm below is O(1).

```
for (i = 0; i < 24; ++i) {
   if (timeHour < 6) {
      tollSchedule[i] = 1.55
   }
   else if (timeHour < 10) {
      tollSchedule[i] = 4.65
   }
   else if (timeHour < 18) {
      tollSchedule[i] = 2.35
   }
   else {
      tollSchedule[i] = 1.55
   }
}
```

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- True
- False

Runtime analysis of nested loops

Runtime analysis for nested loops requires summing the runtime of the inner loop over each outer loop iteration. The resulting summation can be simplified to determine the big-O notation.

PARTICIPATION ACTIVITY 2.6.5: Runtime analysis of nested loop: Selection sort algorithm.

```
for (i = 0; i < N; ++i) {
   indexSmallest = i

   for (j = i + 1; j < N; ++j) {
      if (numbers[j] < numbers[indexSmallest]) {
         indexSmallest = j
      }
   }

   temp = numbers[i]
   numbers[i] = numbers[indexSmallest]
   numbers[indexSmallest] = temp
}
```

4 operations, generalized as constant c

Each of the N outer loop iterations executes 5 operations, generalized as constant d

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$$f(N) = c\left((N - 1) + (N - 2) + \cdots + 2 + 1 + 0\right) = c\left(\frac{N(N - 1)}{2}\right) + d \cdot N$$

$$O(f(N)) = O\left(\frac{c}{2}(N^2 - N) + d \cdot N\right) = O(N^2 + N) = O(N^2)$$

Animation content:

Static Figure:
Begin code:

```
for (i = 0; i < N; ++i) {
  indexSmallest = i

  for (i + 1; j < N; ++j) {
    if (numbers[j] < numbers[indexSmallest]) {
      indexSmallest = j
    }
  }

  temp = numbers[i]
  numbers[i] = numbers[indexSmallest]
  numbers[indexSmallest] = temp
}
```

End code.
The code block is shown with different parts highlighted by various boxes. The following are highlighted in green boxes: the code line "indexSmallest = i", the statement"j = i + 1 " in the header of the inner for loop line, "for (j = i + 1; j < N; ++j)", and the lines
"temp = numbers[i]
numbers[i] = numbers[indexSmallest]
numbers[indexSmallest] = temp". The last lines of code in the green box are labeled, "Each of the N outer loop iterations executes 5 operations, generalized as constant d".
The code lines, "j < N; ++j " within the inner for loop header, "for (j = i + 1; j < N; ++j)", and also the lines "if (numbers[j] < numbers[indexSmallest]) {
indexSmallest = j
}" are in a blue box. The latter lines of code are labeled, "4 operations, generalized as constant c".
Two formulas below the code block, indicating computations for time complexity determination are shown, "f(N) = c((N-1) + (N-2) + ... + 2 + 1 + 0) = c((N(N-1))/2) + d*N" and "O(f(N)) = O((c/2)(N^2 - N) + d*N) = O(N^2 + N) = O(N^2)".

Step 1: For each iteration of the outer loop, the runtime of the inner loop is determined and added together to form a summation. For iteration i = 0, the inner loop executes N - 1 iterations.
The main outer for loop code line, "for (i = 0; i < N; ++i) {", is highlighted. The text, "i = 0", is shown next to the main outer for loop code line. The highlight moves to the inner for loop code line, "for (j = i + 1; j < N; ++j) { ". The text, "j = 1 to N -1", is shown above the inner for loop code line. The text, "N - 1 iterations", is shown next to the line of code. The start of function f(N) is constructed: f(N) = (N - 1) +

Step 2: For i = 1, the inner loop iterates N - 2 times: iterating from j = 2 to j = N - 1.
The highlight moves back to the main outer for loop code line, "for (i = 0; i < N; ++i) {". The text, "i = 1",

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is shown next to the main outer for loop code line. The highlight moves to the inner for loop code line, "for (int j = i + 1; j < N; ++j) { ". The text, "j = 2 to N -1", is shown above the inner for loop code line. The text, "N - 2 iterations", is shown next to the line of code. Additions are added to the function f(N): f(N) = (N - 1) + (N - 2) + … +

Step 3: For i = N - 3, the inner loop iterates twice: iterating from j = N - 2 to N - 1. For i = N - 2, the inner loop iterates once: iterating from j = N - 1 to N - 1.
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The highlight moves back to the main outer for loop code line, "for (i = 0; i < N; ++i) {". The text, "i = N - 3", is shown next to the main outer for loop code line. The highlight moves to the inner for loop code line, "for (j = i + 1; j < N; ++j){ ". The text, "j = N - 2 to N -1", is shown above the inner for loop code line. The text, "2 iterations", is shown next to the inner for loop code line. Additions are added to the function f(N): f(N) = (N - 1) + (N - 2) + … + 2
Once again, the highlighted blue box moves to the main outer for loop code line, "for (i = 0; i < N; ++i) {". The text, "i = N - 3", is shown next to the outer for loop code line. The highlighted box moves to the inner for loop code line, "for (j = i + 1; j < N; ++j){ ". The text, "j = N - 1 to N - 1", is shown above the inner for loop code line. The text, "1 iterations", is shown next to the inner for loop code line. Additions are added to the function f(N): f(N) = (N - 1) + (N - 2) + … + 2 + 1

Step 4: For i = N - 1, the inner loop iterates 0 times. The summation is the sum of a consecutive sequence of numbers from N - 1 to 0.
Additions are added to the function f(N): f(N) = (N - 1) + (N - 2) + … + 2 + 1 + 0

Step 5: The sequence contains N / 2 pairs, each summing to N - 1, and can be simplified.
The function f(N) is simplified. From the first and last parts of the equations; (N - 1) + 0 = N -1. To the second and second to last parts; (N - 2) + 1 = N - 1. To the third and third from last parts; (N- 3) + 2 = N -1. The conclusion is that f(N) = N(N-1)/2.

Step 6: Each iteration of the loops requires a constant number of operations, which is defined as the constant c.
The code lines, " j < N; ++j" within the inner for loop code line, "for (j = i + 1; j < N; ++j) {", and "if (numbers[j] < numbers[indexSmallest]) { indexSmallest = j } ", within the body of the inner for loop, are highlighted in a blue box with the label "4 operations, generalized as constant c". The constant c is added to the f(N) function: f(N): f(N) = c((N - 1) + (N - 2) + … + 2 + 1 + 0) = c(N(N-1)/2)

Step 7: Additionally, each iteration of the outer loop requires a constant number of operations, which is defined as the constant d.
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The code lines, "indexSmallest = i; ", "j = i + 1;", within the inner for loop code line, "for (j = i + 1; j < N; ++j) {", and "temp = numbers[i]", "numbers[i] = numbers[indexSmallest]", "numbers[indexSmallest] = temp" are highlighted in a green box. The last lines of code highlighted in green are labeled "Each of the N outer loop iterations executes 5 operations, generalized as constant d". The constant d is added to the f(N) function so it reads: f(N): f(N) = c((N - 1) + (N - 2) + … + 2 + 1 + 0) = c(N(N-1)/2) + d·N

Step 8: Big O notation omits the constant values, and the runtime is equal to the summation of the total inner-loop iterations.
The function O(f(N)) is shown: O(f(N)) = O(c/2 (N^2 - N) + d·N) = O(N^2 + N) = O(N^2)

Animation captions:

1. For each iteration of the outer loop, the runtime of the inner loop is determined and added together to form a summation. For iteration i = 0, the inner loop executes N - 1 iterations.
2. For i = 1, the inner loop iterates N - 2 times: iterating from j = 2 to N - 1.
3. For i = N - 3, the inner loop iterates twice: iterating from j = N - 2 to N - 1. For i = N - 2, the inner loop iterates once: iterating from j = N - 1 to N - 1.
4. For i = N - 1, the inner loop iterates 0 times. The summation is the sum of a consecutive sequence of numbers from N - 1 to 0.
5. The sequence contains N / 2 pairs, each summing to N - 1, and can be simplified.
6. Each iteration of the loops requires a constant number of operations, which is defined as the constant c.
7. Additionally, each iteration of the outer loop requires a constant number of operations, which is defined as the constant d.
8. Big-O notation omits the constant values, and the runtime is equal to the summation of the total inner loop iterations.

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Figure 2.6.1: Common summation: Summation of consecutive numbers.

$$(N-1)+(N-2)+\cdots+2+1=\frac{N(N-1)}{2}=O(N^2)$$

PARTICIPATION ACTIVITY 2.6.6: Nested loops.

Determine the big-O worst-case runtime for each algorithm.

1)
```
for (i = 0; i < N; i++)  {
   for (j = 0; j < N; j++) {
      if (numbers[i] <
numbers[j]) {
         ++eqPerms
      }
      else {
         ++neqPerms
      }
   }
}
```

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- O(N)
- $O(N^2)$

2)
```
for (i = 0; i < N; i++)  {
   for (j = 0; j < (N - 1);
j++) {
      if (numbers[j + 1] <
numbers[j]) {
         temp = numbers[j]
         numbers[j] = numbers[j
+ 1]
         numbers[j + 1] = temp
      }
   }
}
```

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- O(N)
- $O(N^2)$

3)
```
for (i = 0; i < N; i = i + 2)
{
   for (j = 0; j < N; j = j +
2) {
      cVals[i][j] = inVals[i] *
j
   }
}
```

- O(N)
- $O(N^2)$

4)
```
for (i = 0; i < N; ++i)  {
   for (j = i; j < N - 1; ++j)
{
      cVals[i][j] = inVals[i] *
j
   }
}
```

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- $O(N^2)$
- $O(N^3)$

5)
```
for (i = 0; i < N; ++i) {
   sum = 0
   for (j = 0; j < N; ++j) {
      for (k = 0; k < N; ++k) {
         sum = sum + aVals[i]
[k] * bVals[k][j]
      }

      cVals[i][j] = sum
   }
}
```

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- O(N)
- $O(N^2)$
- $O(N^3)$

2.7 Recursive definitions

Recursive algorithms

An ***algorithm*** is a sequence of steps, including at least 1 terminating step, for solving a problem. A ***recursive algorithm*** is an algorithm that breaks the problem into smaller subproblems and applies the algorithm itself to solve the smaller subproblems.

Because a problem cannot be endlessly divided into smaller subproblems, a recursive algorithm must have a ***base case***: A case where a recursive algorithm completes without applying itself to a smaller subproblem.

PARTICIPATION ACTIVITY 2.7.1: Recursive factorial algorithm for positive numbers.

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Factorial(N):

```
if (N == 1)
   return 1
```
Base case

```
else
   return N * Factorial(N - 1)
```
Non-base case

Recursive logic

Animation content:

Static figure: A recursive algorithm labeled Factorial(N): containing two code blocks is displayed. The first code block is labeled Base case and the second code block is labeled Non-base case.
Base case:
Begin pseudocode:

```
if (N == 1) {
  return 1
}
```

End pseudocode.
Non-base case:
Begin pseudocode:

```
else {
  return N * Factorial(N - 1)
}
```

End pseudocode.
Step 1: A recursive algorithm to compute N factorial has two parts: the base case and the non-base case.
Step 2: N is assumed to be a positive integer. The base case is when N equals 1, wherein a result of 1 is returned.
Step 3: The non-base case computes the result by multiplying N by (N - 1) factorial.
Step 4: The algorithm applying itself to a smaller subproblem is what makes the algorithm recursive. The code, Factorial(N - 1), is highlighted and the text, Recursive logic, appears below the code.

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Animation captions:

1. A recursive algorithm to compute N factorial has 2 parts: the base case and the non-base case.
2. N is assumed to be a positive integer. The base case is when N equals 1, wherein a result of 1 is returned.
3. The non-base case computes the result by multiplying N by (N - 1) factorial.
4. The algorithm applying itself to a smaller subproblem is what makes the algorithm recursive.

PARTICIPATION ACTIVITY 2.7.2: Recursive algorithms.

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1) A recursive algorithm applies itself to a smaller subproblem in all cases.
   - ○ True
   - ○ False

2) The base case is what ensures that a recursive algorithm eventually terminates.

- True
- False

3) The presence of a base case is what identifies an algorithm as being recursive.

- True
- False

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Recursive functions

A ***recursive function*** is a function that calls itself. Recursive functions are commonly used to implement recursive algorithms.

Table 2.7.1: Sample recursive functions: Factorial, CumulativeSum, and ReverseList.

```
Factorial(N) {
   if (N == 1)
      return 1
   else
      return N * Factorial(N - 1)
}
```

```
CumulativeSum(N) {
   if (N == 0)
      return 0
   else
      return N + CumulativeSum(N - 1)
}
```

```
ReverseList(list, startIndex, endIndex) {
   if (startIndex >= endIndex)
      return
   else {
      Swap elements at startIndex and endIndex
      ReverseList(list, startIndex + 1, endIndex - 1)
   }
}
```

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PARTICIPATION ACTIVITY 2.7.3: CumulativeSum recursive function.

1) What is the condition for the base case in the CumulativeSum function?

- N equals 0
- N does not equal 0

2) If Factorial(6) is called, how many additional calls are made to Factorial to compute the result of 720?

- 7
- 5
- 3

3) Suppose ReverseList is called on a list of size 3, a start index of 0, and an out-of-bounds end index of 3. The base case ensures that the function still properly reverses the list.

- True
- False

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2.8 Recursive algorithms

Fibonacci numbers

The ***Fibonacci sequence*** is a numerical sequence where each term is the sum of the previous 2 terms in the sequence, except the first 2 terms, which are 0 and 1. A recursive function can be used to calculate a ***Fibonacci number***: A term in the Fibonacci sequence.

Figure 2.8.1: FibonacciNumber recursive function.

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```
FibonacciNumber(termIndex) {
   if (termIndex == 0)
      return 0
   else if (termIndex == 1)
      return 1
   else
      return FibonacciNumber(termIndex - 1) + FibonacciNumber(termIndex
- 2)
}
```

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PARTICIPATION ACTIVITY 2.8.1: FibonacciNumber recursive function.

1) What does FibonacciNumber(2) return?

Check Show answer

2) What does FibonacciNumber(4) return?

Check Show answer

3) What does FibonacciNumber(8) return?

Check Show answer

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Recursive binary search

Binary search is an algorithm that searches a sorted list for a key by first comparing the key to the middle element in the list and recursively searching half of the remaining list so long as the key is not found.

Binary search first checks the middle element of the list. If the search key is found, the algorithm returns the index. If the search key is not found, the algorithm recursively searches the remaining left

sublist (if the search key was less than the middle element) or the remaining right sublist (if the search key was greater than the middle element).

Figure 2.8.2: BinarySearch recursive algorithm.

```
BinarySearch(numbers, low, high, key) {
   if (low > high)
      return -1

   mid = (low + high) / 2
   if (numbers[mid] < key) {
      return BinarySearch(numbers, mid + 1, high,
key)
   }
   else if (numbers[mid] > key) {
      return BinarySearch(numbers, low, mid - 1,
key)
   }
   return mid
}
```

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PARTICIPATION ACTIVITY 2.8.2: Recursive binary search.

Suppose BinarySearch(numbers, 0, 6, 42) is used to search the list (14, 26, 42, 59, 71, 88, 92) for key 42.

1) What is the first middle element that is compared against 42?

- 42
- 59
- 71

2) What will the low and high argument values be for the first recursive call?

- low = 0
  high = 2
- low = 0
  high = 3
- low = 4
  high = 6

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3) How many calls to BinarySearch will be made by the time 42 is found?

- 2
- 3
- 4

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PARTICIPATION ACTIVITY 2.8.3: Recursive binary search base case.

1) Which does not describe a base case for BinarySearch?

- The low argument is greater than the high argument.
- The list element at index `mid` equals the key.
- The list element at index `mid` is less than the key.

2.9 Analyzing the time complexity of recursive algorithms

Recurrence relations

The runtime complexity T(N) of a recursive function will have function T on both sides of the equation. Ex: Binary search performs constant time operations, then a recursive call that operates on half of the input, making the runtime complexity T(N) = O(1) + T(N / 2). Such a function is known as a ***recurrence relation***: A function f(N) that is defined in terms of the same function operating on a value < N.

Using O-notation to express runtime complexity of a recursive function requires solving the recurrence relation. For simpler recursive functions such as binary search, runtime complexity can be determined by expressing the number of function calls as a function of N.

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PARTICIPATION ACTIVITY 2.9.1: Worst case binary search runtime complexity.

```
BinarySearch(numbers, low, high, key) {
   if (low > high) {
      return -1
   }
   mid = (low + high) / 2
   if (numbers[mid] < key) {
      return BinarySearch(numbers, mid + 1, high, key)
   }
   else if (numbers[mid] > key) {
      return BinarySearch(numbers, low, mid - 1, key)
   }
   return mid
}
```

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Input size	Number of function calls
1	1
2	2
4	3
8	4
16	5
32	6
...	...
N	$\log_2 N + 1$

```
If: T(N) = O(1) + T(N / 2)
Then: T(N) = O(log N)
```

Animation content:

Static figure: A code block and a table with two columns, left is "Input size" and right is "Number of function calls", are displayed.
Begin pseudocode:

```
BinarySearch(numbers, low, high, key) {
  if (low > high) {
    return -1
  }
  mid = (low + high) / 2
  if (numbers[mid] < key) {
    return BinarySearch(numbers, mid + 1, high, key)
  }
  else if (numbers[mid] > key) {
    return BinarySearch(numbers, low, mid - 1, key)
  }
  return mid
}
```

End pseudocode.

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Table data:
1, 1

2, 2
4, 3
8, 4
16, 5
32, 6
..., ...
N, log_2(N) + 1
End table data.

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Below the table is the final conclusion about time complexity:
If: T(N) = O(1) + T(N/2)
Then: T(N) = O(log N)

Animation captions:

1. In the non-base case, BinarySearch does some O(1) operations plus a recursive call on half the input list.
2. The maximum number of recursive calls can be computed for any known input size. For size 1, 1 recursive call is made.
3. Additional entries in the table can be filled. A list of size 32 is split in half 6 times before encountering the base case.
4. By analyzing the pattern, the total number of function calls can be expressed as a function of N.
5. The number of function calls corresponds to the runtime complexity.

PARTICIPATION ACTIVITY 2.9.2: Binary search and recurrence relations.

1) When the low and high arguments are equal, BinarySearch() has 0 items to search and so immediately returns -1.
   - ○ True
   - ○ False

2) Suppose BinarySearch() is used to search for a key within an array with 64 numbers. If the key is not found, how many recursive calls to BinarySearch() are made?
   - ○ 1
   - ○ 7
   - ○ 64

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3) Which function is a recurrence relation?

- $T(N) = N^2 + 6N + 2$
- $T(N) = 6N + T(N/4)$
- $T(N) = log_2 N$

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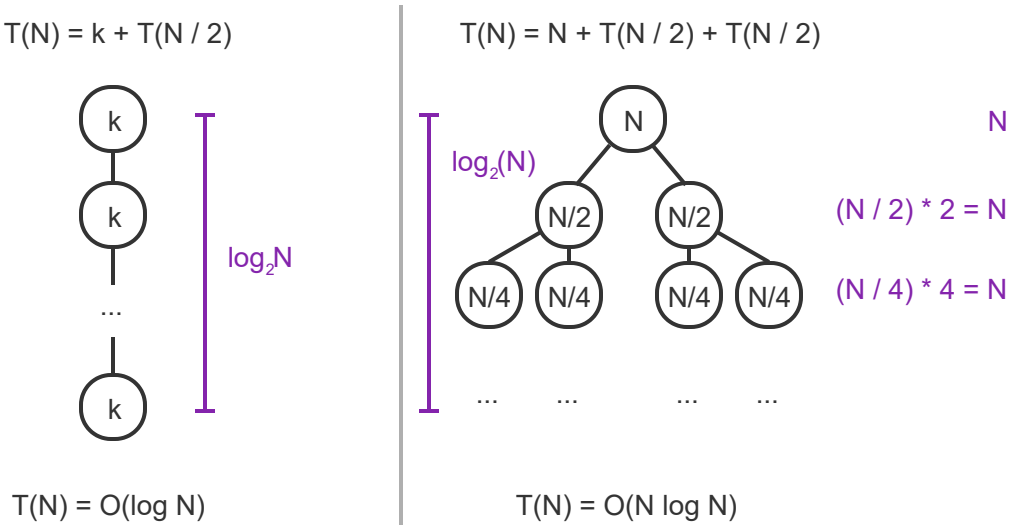
Recursion trees

The runtime complexity of any recursive function can be split into 2 parts: operations done directly by the function and operations done by recursive calls made by the function. Ex: For binary search's T(N) = O(1) + T(N / 2), O(1) represents operations directly done by the function and T(N / 2) represents operation done by a recursive call. A useful tool for solving recurrences is a ***recursion tree***: A visual diagram of an operation done by a recursive function, that separates operations done directly by the function and operations done by recursive calls.

PARTICIPATION ACTIVITY 2.9.3: Recursion trees.



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Animation content:

Static figure:
On left, equation at top: T(N) = k + T(N / 2). Below equation is a visual structure for the recursion tree. Atop the tree is the text "k" in a circle. The circle is known as a "node". A line below connects to another "k" node. A line below heads toward "..." text, then another line that connects to another "k" node at the bottom. Height bar appears to the side of the recursion tree indicating the height:

log_2(N).

On right, equation at top: T(N) = T(N / 2) + T(N / 2). Below equation is a visual structure for the recursion tree. Atop the tree is a node with text "N". Two lines below the "N" node connect to "N / 2" nodes. Each of those has two lines below connecting to "N / 4" nodes. "..." labels below each "N / 4" node indicate the the process continues. Height bar appears to the side of the recursion tree indicating the height: log_2(N). Each level of nodes has a calculation to the side that equates to N: top level is "N", next level is "(N / 2) * 2 = N", and last level is "(N / 4) * 4 = N".

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Animation captions:

1. An algorithm like binary search does a constant number of operations, k, followed by a recursive call on half the array.
2. The root node in the recursion tree represents k operations inside the first function call.
3. Recursive operations are represented below the node. The first recursive call also does k operations.
4. The tree's height corresponds to the number of recursive calls. Splitting the input in half each time results in $log_2 N$ recursive calls. O($log_2 N$) = O($log N$).
5. Another algorithm may perform N operations then two recursive calls, each on N / 2 items. The root node represents N operations.
6. The initial call makes two recursive calls, each of which has a local N value of the initial N value / 2.
7. N operations are done per level.
8. The tree has O($log_2 N$) levels. O($N * log_2 N$) = O($N log N$) operations are done in total.

PARTICIPATION ACTIVITY 2.9.4: Matching recursion trees with runtime complexities.

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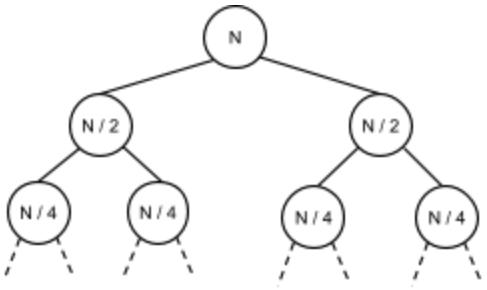
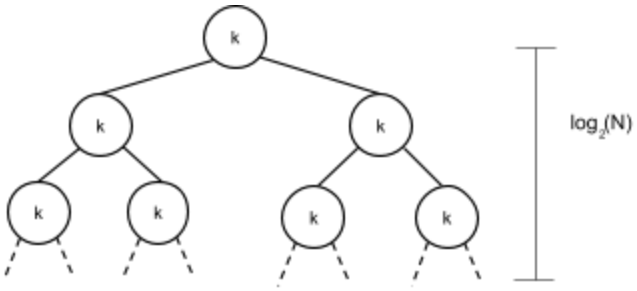


Tree 1



Tree 2

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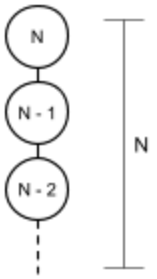
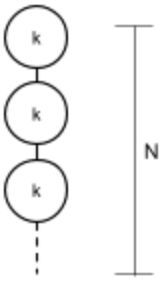


Tree 3



Tree 4

If unable to drag and drop, refresh the page.

Tree 3 | Tree 4 | Tree 1 | Tree 2

	T(N) = k + T(N / 2) + T(N / 2)
	T(N) = k + T(N - 1)
	T(N) = N + T(N - 1)
	T(N) = N + T(N / 2) + T(N / 2)

Reset

PARTICIPATION ACTIVITY 2.9.5: Recursion trees.

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Suppose a recursive function's runtime is $T(N) = 7 + T(N - 1)$.

1) How many levels will the recursion tree have?

- 7

- $log_2 N$
- N

2) What is the runtime complexity of the function using O notation?

- O(1)
- O($logN$)
- O(N)

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PARTICIPATION ACTIVITY 2.9.6: Recursion trees.

Suppose a recursive function's runtime is $T(N) = N + T(N - 1)$.

1) How many levels will the recursion tree have?

- $log_2 N$
- N
- N^2

2) The runtime can be expressed by the series N + (N - 1) + (N - 2) + ... + 3 + 2 + 1. Which expression is mathematically equivalent?

- $N * log_2 N$
- $(N/2) * (N + 1)$

3) What is the runtime complexity of the function using Big O notation?

- O(N)
- O(N^2)

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